Physics 3143 – Quantum Mechanics – Homework #6

1. Show that the Hermite functions are orthogonal for (a) $n=1$ and $m=2$; (b) $n=1$ and $m=3$.

2. Show that the Legendre functions are normalized for $n=0,1,2$.

3. Show that the associated Legendre functions, $P_1^1(x)$ and $P_2^1(x)$ are orthogonal.

4. Show that the Laguerre function for $n=1$ is normalized.

5. For the description of the harmonic oscillator within the Dirac framework, prove that $a|n\rangle$ is an eigenvector of $\hat{H}$ with eigenvalue $(E_n - \hbar\omega)$ where a is the lowering operator

6. Show that $[\hat{\mathcal{L}}_+, \hat{\mathcal{L}}_-] = 2\hbar \hat{L}_z$

7. Show that if $\hat{\mathcal{L}}_z |\ell m\rangle = m\hbar |\ell m\rangle$, then $\hat{\mathcal{L}}_z (\hat{\mathcal{L}}_- |\ell m\rangle) = (m-1)\hbar (\hat{\mathcal{L}}_- |\ell m\rangle)$

8. Show that

$$\hat{\mathcal{L}}_x = -i\hbar\left(-\sin\phi\, \partial/\partial\theta - \cot\theta \cos\phi\, \partial/\partial\phi\right)$$
$$\hat{\mathcal{L}}_- = -\hbar e^{-i\phi}\left(\partial/\partial\theta - i\cot\theta\, \partial/\partial\phi\right)$$

9. From the relation

$$|c_{\ell\ell}|^2\, 2\pi \int_0^{\pi} \sin^{2\ell+1}\theta\, d\theta = 1$$

where $c_{\ell\ell}$ is the normalization constant and where $\psi_{\ell\ell}(\theta,\phi) = c_{\ell\ell} \sin^{\ell}\theta \, e^{i\ell\phi}$,

show that $c_{\ell\ell} = ((2\ell+1)!)^{1/2} / 2^{\ell}\ell! \, (4\pi)^{1/2}$

10. Using the point mass approximation for the moment of inertia

$$I = \sum_i m_i r_i^2$$

where the m_i are the masses of the constituent particles and the r_i are their distances from the center of mass, determine the distance of separation of the hydrogen and chlorine nuclei in an HCl molecule. Use $m(H) = 1.67357 \times 10^{-27}$ kg, $m(^{35}Cl) = 5.80682 \times 10^{-26}$ kg, $m(^{37}Cl) = 6.13845 \times 10^{-26}$ kg. Calculate the separation in $H^{35}Cl$ and in $H^{37}Cl$ (in Å) for $v=0$ and $v=1$ from the information provided in the handout on molecular structure and rotations.
Recall $E = h\nu = hc/\lambda$ whence $E/hc = 1/\lambda$ permits the definition of energy in units of length^{-1} or cm^{-1}.

11. Show that the raising operator a^{+} in the Dirac framework (H.O.) can be written in the form $-d/dq + q$ obtained using the factorization method.

12. From the bound on the energy, E, $E \geq 0$ in the harmonic oscillator model of Dirac, there must be a lowest energy eigenstate $|n_0\rangle$ whence

$$a|n_0\rangle = 0 \qquad \text{(see class notes)}$$

Show that this expression can be written using the position representation to obtain an expression in the form of the factorization expression

$$\left(d/dq + q \right) \psi_0(q) = 0$$

13. Calculate <r> for a 2s electron in a hydrogen atom

Given $\int_0^\infty x^n e^{-ax}\, dx = n!/a^{n+1}$ $\quad n > -1, \; a > 0$

14. For the H atom, show that ψ_{1s} and ψ_{2s} are orthogonal

15. Please work through problem 5.1 in Griffiths